THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4250 Game Theory, 2019-2020 Term 1
Mid-term Examination
Time allowed: 90 mins
Answer all questions. Full marks: 60

1. (10 marks) In the less than half game, there is one pile of chips on the table. In each turn, a player may remove k chips where k is a positive integer less than half of the number of chips remaining. (If there are 10 chips remaining on the table, a player may remove at least 1 chip and at most 4 chips.) The game ends if there are 1 or 2 chips remaining and the player who makes the last move wins. Let $g(x)$ be the Sprague-Grundy function of the less than half game.
   (a) Find $g(6), g(7), g(8), g(9)$.
   (b) Make a guess of the set of P-positions of the less than half game.
   (c) Prove your assertion in (b).

   Solution:
   (a) $g(6) = 1, g(7) = 3, g(8) = 0, g(9) = 4$.
   (b) P-positions: $\{x : x = 0 \text{ or } 2^k, k = 0, 1, 2, 3 \cdots\}$.
   (c) Standard steps in lecture notes.

2. (10 marks) In the Wythoff's game, there are 2 piles of chips. In each turn, a player may either remove any positive number of chips from one of the piles, or remove the same positive number of chips from both piles. The player removing the last chip wins.
   (a) If $(a, 18)$ is a P-position, find a.
   (b) If $(b, b + 32)$ is a P-position, find b.
   (c) Find all winning moves from the positions $(18, 27)$ and $(120, 150)$.

   Solution:
   (a) $\lfloor n\varphi \rfloor + n = 18$, then $n = 7$. Hence, $a = \lfloor n\varphi \rfloor = 11$.
   (b) $n = 32$, $b = \lfloor n\varphi \rfloor = 51$.
   (c) For $(18, 27)$: $(18, 11)$ and $(14, 23)$;
   For $(120, 150)$: $(74, 120)$ and $(48, 78)$.

3. (10 marks) Consider the following three games.

   Game 1: More than one game (In each turn, a player removes more than 1 chip. The terminal positions are $0, 1$.)

   Game 2: Subtraction game with subtraction set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

   Game 3: 1-pile nim

Let g_1, g_2, g_3 be the Sprague-Grundy functions of the three games respectively. Let G be the sum of the three games and g be the Sprague-Grundy function of G.

(a) Find $g_1(12), g_1(13), g_1(14)$.

(b) Find $g(19, 16, 6)$.

(c) Find all winning moves of G from the position $(19, 16, 6)$.

Solution:

(a)

x	0	1	2	3	4	5	6	7	8	9	10	$\cdots$
$g_1(x)$	0	0	1	1	2	2	3	3	4	4	5	$\cdots$

$g(12) = 6, g(13) = 6, g(14) = 7.$

(b)

$$\begin{aligned} & g(19,16,6) \\ &= g_1(19) \oplus g_2(16) \oplus g_3(6) \\ &= 9 \oplus 5 \oplus 6 \\ &= 10 \end{aligned}$$

(c)

(i) If $(19, 16, n)$ is a winning move, $g_1(19) \oplus g_2(16) = 12 > 6$. This case is rejected.

(ii) If $(n, 16, 6)$ is a winning move, $g_2(16) \oplus g_3(6) = 3 > 6$. Hence $g_1(n) = 3$ and $n \leq 19$. Thus, $n = 6$ or 7.

(iii) If $(19, n, 6)$ is a winning move, $g_1(19) \oplus g_3(6) = 15$. However, $g_3(n) \neq 15 \forall n \in \mathbb{Z}^*$. This case is rejected.

In all, winning moves are (6, 16, 6) or (7, 16, 6).

4. (10 marks) Let

$$A = \begin{pmatrix} 1 & 2 & -3 & 4 & 0 \\ 2 & 3 & -1 & 3 & 1 \\ 0 & -2 & 4 & 1 & 2 \end{pmatrix}$$

(a) Write down the reduced matrix obtained by deleting all dominated rows and columns of A.

(b) Use the reduced matrix to solve the game matrix A.

Solution:

(a)

$$A = \begin{pmatrix} 1 & 2 & -3 & 4 & 0 \\ 2 & 3 & -1 & 3 & 1 \\ 0 & -2 & 4 & 1 & 2 \end{pmatrix}$$

$$\Longrightarrow A_1 = \begin{pmatrix} 1 & 2 & -3 & 0 \\ 2 & 3 & -1 & 1 \\ 0 & -2 & 4 & 2 \end{pmatrix}$$

$$\Longrightarrow A_2 = \begin{pmatrix} 2 & 3 & -1 & 1 \\ 0 & -2 & 4 & 2 \end{pmatrix}.$$

(b) $c_1 : v = 2x$
$c_2 : v = 3x - 2(1-x)$
$c_3 : v = -x + 4(1-x)$
$c_5 : v = x + 2(1-x)$
Draw the graph, find the maximum point of the lower envelope: $x = 0.6, v = 1$.
Hence for the game matrix A,
the value of A is 1,
the maximin strategy $\vec{p} = (0, 0.6, 0.4)$,
the minimax strategy $\vec{q} = (0, 0.5, 0.5, 0, 0)$.

5. (10 marks) Use simplex method to solve the game matrix

$$\begin{pmatrix} 1 & 4 & -1 \\ 2 & -1 & 0 \\ 0 & -2 & 1 \end{pmatrix}.$$

Solution:
Step 1. Add 2 to each entry, we get

$$\begin{pmatrix} 3 & 6 & 1 \\ 4 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix}.$$

Step 2. Set up the tableau as

	y_1	y_2	y_3	
x_1	3	6	1	1
x_2	4	1	2	1
x_3	2	0	3	1
	1	1	1	0

Step 3. Apply pivoting operations, we have

	y_1	y_2	y_3	
x_1	3	6	1	1
x_2	4^*	1	2	1
x_3	2	0	3	1
	1	1	1	0

$\rightarrow$

	x_2	y_2	y_3	
x_1	$-\frac{3}{4}$	$\frac{21^*}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$
y_1	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$
x_3	$-\frac{1}{2}$	$-\frac{1}{2}$	2	$\frac{1}{2}$
	$-\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{2}$	$-\frac{1}{4}$

$\rightarrow$

	x_2	x_1	y_3	
y_2	$-\frac{1}{7}$	$\frac{4}{21}$	$-\frac{2}{21}$	$\frac{1}{21}$
y_1	$\frac{2}{7}$	$-\frac{1}{21}$	$\frac{11}{21}$	$\frac{5}{21}$
x_3	$-\frac{4}{7}$	$\frac{2}{21}$	$\frac{41^*}{21}$	$\frac{11}{21}$
	$-\frac{1}{7}$	$-\frac{1}{7}$	$\frac{4}{7}$	$-\frac{2}{7}$

$\rightarrow$

	x_2	x_1	x_3	
y_2	$-\frac{7}{41}$	$\frac{8}{41}$	$-\frac{2}{41}$	$\frac{3}{41}$
y_1	$\frac{18^*}{41}$	$-\frac{3}{41}$	$-\frac{11}{41}$	$\frac{4}{41}$
y_3	$-\frac{12}{41}$	$\frac{2}{41}$	$\frac{41}{21}$	$\frac{11}{41}$
	$\frac{1}{41}$	$-\frac{7}{41}$	$-\frac{12}{41}$	$-\frac{18}{41}$

$\rightarrow$

	y_1	x_1	x_3	
y_2	$\frac{7}{18}$	$\frac{1}{6}$	$-\frac{1}{18}$	$\frac{1}{9}$
x_2	$\frac{41}{18}$	$-\frac{1}{6}$	$-\frac{11}{18}$	$\frac{2}{9}$
y_3	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$
	$-\frac{1}{18}$	$-\frac{1}{6}$	$-\frac{5}{18}$	$-\frac{4}{9}$

Hence $d = \frac{4}{9}, \vec{p} = (\frac{3}{8}, 0, \frac{5}{8})^T, \vec{q} = (0, \frac{1}{4}, \frac{3}{4})^T$, and the value of the game is $\frac{1}{4}$.

6. (10 marks) Let n be a positive integer and

$$A = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n-1} & \frac{1}{n} \\ 1 & 0 & \frac{1}{3} & \cdots & \frac{1}{n-1} & \frac{1}{n} \\ 1 & \frac{1}{2} & 0 & \cdots & \frac{1}{n-1} & \frac{1}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & 0 & \frac{1}{n} \\ 1 & \frac{1}{2} & \frac{1}{3} & \cdots & \frac{1}{n-1} & 0 \end{pmatrix}.$$

(a) Find a vector $\mathbf{y} \in \mathbb{R}^n$ such that $A\mathbf{y}^T = (n-1, n-1, \cdots, n-1)^T$.

(b) Find a maximin strategy for the row player, a minimax strategy for the column player and the value of A.

Solution:

(a) $\vec{y} = (1, 2, \cdots, n)^T, i.e. y_i = i, 1 \leq i \leq n.$

(b) By using Principle of Indifference, we have

$$\begin{cases} p_2 + p_3 + \cdots + p_n = v \\ \frac{1}{2}p_1 + \frac{1}{2}p_3 + \cdots + \frac{1}{2}p_n = v \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{1}{n}p_1 + \frac{1}{n}p_2 + \cdots + \frac{1}{n}p_{n-1} = v \end{cases}$$

then,

$$\begin{cases} p_1 + p_2 + p_3 + \cdots + p_n = v + p_1 \\ p_1 + p_2 + p_3 + \cdots + p_n = 2v + p_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_1 + p_2 + \cdots + p_{n-1} + p_n = nv + p_n \end{cases}$$

Since $p_1 + p_2 + \cdots + p_n = 1$, then

$$\begin{cases} v + p_1 = 1 \\ 2v + p_2 = 1 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ nv + p_n = 1. \end{cases}$$

We can get $(1 + 2 + 3 + \cdots + n)v + (p_1 + p_2 + \cdots + p_n) = n$.
Thus $v = \frac{n-1}{1+2+3+\cdots+n} = \frac{2n-2}{n^2+n}$, and $p_i = 1 - iv = 1 - \frac{2n-2}{n^2+n}i, i = 1, 2, 3, \cdots, n$. Hence $\vec{p} = (p_1, p_2, \cdots, p_n)$.
Similarly, $\vec{q} = (q_1, q_2, \cdots, q_n), q_i = \frac{2i}{n^2+n}, 1 \le i \le n$.